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COUNTING ROOTED AND UNROOTED TRIANGULAR MAPS

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Abstract. In this paper, we describe a new way to count isomorphism classes of rooted triangular maps and unrooted triangular maps. We point out an explicit connection with the asymptotic expansion of the Airy function. The analysis presented here is used in a recent paper "*Vidal (2007)*" to present an algorithm that generates in optimal amortized time an exhaustive list of triangular maps of a given size.

Keywords. rooted triangular maps, unrooted triangular maps, generating functions, Airy funtion, cycle index series.

Introduction

Triangulations of surfaces constitute an important data structure in computer graphics as they provide a handy discrete model of surfaces. It has proven invaluable for instance to model the shape of objects in computed graphics. From the point of view of computer science, the applications of surface triangulations are well known and numerous, they touch both practical and theoretical aspects of the discipline and they range from computer graphics to discret methods of solving partial differential equations. They also play a central role in many algorithms of computational geometry, a fast growing subject having an heavy industrial impact as it is used in computer aided design.

One particularly interesting treat of the subject, apart from its broad range of applications, is precisely its ubiquity both in computer science, mathematical physics and even pure mathematics, providing generous range of fruitful exchange between seemingly remote parts of science. From the point of view of mathematics, the theory of combinatorial maps is also a venerable subject dating back to Cayley and Hamilton. Since those times, it generated an impressive amount of results of all sorts concerning the particular enumeration problem of counting the *rooted* combinatorial maps. Those results came from various communities of researchers, each with its own methods and tradition. Among them, *enumerative combinatorists* of course played a significant role, starting with pioneering works by Tutte [17] on rooted planar maps. Those works where at first motivated by the four color problem. *Theoretical physicists* also played a significant role, starting with the work by t'Hooft [16] on integration on random matrix spaces and Feynman diagrams. Pure mathematicians like Harer and Zagier [5] also have contributed to the theory in connection with cutting edge algebraic geometry problems concerning moduli spaces of Riemann surfaces. Last but not least, one must mention in mathematical physics the Witten-Kontsevich model of quantum gravity [7] using in a central fashion the higher combinatorics of triangular maps and trivalent diagrams.

Although *a lot* is known concerning the theory of *rooted* combinatorial maps, *very little* is currently known about the outstanding problem of enumeration of *unrooted* combinatorial maps up to isomorphism except for planar maps with the pioneering work of Liskovets [9]. It appears as a very difficult problem of combinatorics, which stayed barely untouched for almost 150 years. As a matter of fact, the only general result on those important objects were up to now contained in the recent paper by Mednykh and Nedela [14]. In section 2.4 of this paper, we give our first contribution to this problem, namely in the form of a generating series giving the number of *unrooted* triangular maps (*c.f.* series 20 on page 55).

In this paper, a *triangular map* is a triangulation of a (not necessarily connected) oriented surface without boundary, and its *size* is an integer divisible by 6, hence of the form $n = 6k$, such that the triangulation has $2k$ triangular faces and $3k$ edges. Apart from those unrooted map enumeration results the most interesting theorems of this article are the following.

Theorem 1. *Let a_n be the number of* labelled *triangular maps of size n. Then, $a_n = 0$ if n isn't a multiple of 6 and,*

$$a_{6k} = \frac{(6k)!}{k!}\left(\frac{1}{6}\right)_k\left(\frac{5}{6}\right)_k 6^k \tag{1}$$

where $(x)_k = x\,(x+1)\dots(x+k-1)$ is the Pochhammer symbol. Therefore, the exponential generating series of the a_n is hypergeometric and divergent. We have,

$$\sum_{n\geq 0}\frac{a_n}{n!}\,z^n = {}_2\mathrm{F}_0\left(\begin{array}{c}\frac{1}{6},\frac{5}{6}\\ -\end{array}\middle|\, 6\,z^6\right) \tag{2}$$

$$= \sum_{k\geq 0}\frac{(\frac{1}{6})_k(\frac{5}{6})_k}{k!}\,6^k\,z^{6k} \tag{3}$$

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