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STATISTICS ON GRAPHS, EXPONENTIAL FORMULA AND COMBINATORIAL PHYSICS

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Abstract. The concern of this paper is a famous combinatorial formula known under the name "exponential formula". It occurs quite naturally in many contexts (physics, mathematics, computer science). Roughly speaking, it expresses that the exponential generating function of a whole structure is equal to the exponential of those of connected substructures. Keeping this descriptive statement as a guideline, we develop a general framework to handle many different situations in which the exponential formula can be applied.
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1 Introduction

Applying the exponential paradigm one can feel sometimes incomfortable wondering whether "one has the right" to do so (as for coloured structures, for example). The following paper is aimed at giving a rather large framework where this formula holds.

Exponential formula can be traced back to works by Touchard and Ridell & Uhlenbeck [20, 17]. For an other exposition, see for example [4, 7, 9, 19].

We are interested to compute various examples of EGF for combinatorial objects having (a finite set of) nodes (i.e. their set-theoretical support) so we use as central concept the mapping σ which associates to every structure, its set of (labels of its) nodes.
We need to draw what could be called "square-free decomposable objects" (SFD). This version is suited to our needs for the "exponential formula" and it is sufficiently general to contain, as a particular case, the case of multivariate series.

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2 Partial semigroups

Let us call partial semigroup a semigroup with a partially defined associative law (see for instance [6] for usual semigroups and [1, 14, 18] for more details on structures with a partially defined binary operation). More precisely, a *partial semigroup* is a pair $(S, *)$ where S is a set and $*$ is a (partially defined) function $S \times S \to S$ such that the two (again partially defined) functions $S \times S \times S \to S$

$$(x, y, z) \mapsto (x * y) * z \text{ and } (x, y, z) \mapsto x * (y * z) \quad (1)$$

coincide (same domain and values). Using this requirement one can see that the values of the (partially defined) functions $S^n \to S$

$$(x_1, \cdots, x_n) \mapsto E_T(x_1, \cdots, x_n) \quad (2)$$

obtained by evaluating the expression formed by labelling by x_i (from left to right) the ith leaf of a binary tree T with n nodes and by $*$ its internal nodes, is independant of T. We will denote $x_1 * \cdots * x_n$ their common value. In this paper we restrict our attention to *commutative* semigroups. By this we mean that the value $x_1 * \cdots * x_n$ does not depend on the relative order of the x_i. A nonempty partial semigroup $(S, *)$ has a *(two-sided and total) unit* $\epsilon \in S$ if, and only if, for every $\omega \in S$, $\omega * \epsilon = \omega = \epsilon * \omega$. Using associativity of $*$, it can be easily checked that if S has a unit, then it is unique.

Example 2.1. Let F be a set of sets (resp. which contains $\emptyset$ as an element) and which is closed under the disjoint sum $\sqcup$, *i.e.*, if $A, B \in F$ such that $A \cap B = \emptyset$, then $A \cup B (= A \sqcup B) \in F$. Then $(F, \sqcup)$ is a partial semigroup (resp. partial semigroup with unit).

3 Square-free decomposable partial semigroups

Let $2^{(\mathbb{N}^+)}$ be the set of all finite subsets of the positive integers $\mathbb{N}^+$ and $(S, \oplus)$ be a partial semigroup with unit (here denoted ϵ) equipped with a mapping $\sigma : S \to 2^{(\mathbb{N}^+)}$, called the *(set-theoretic) support mapping*. Let D be the domain of $\oplus$. The triple $(S, \oplus, \sigma)$ is called *square-free decomposable* (SFD) if, and only if, it fulfills the two following conditions.