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EXISTENCE OF SOLUTIONS TO A NONLINEAR PROBLEM INVOLVING ISOTROPIC DEFORMATIONS

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Abstract. In this paper we show, under suitable hypotheses on the boundary datum φ, existence of Lipschitz maps $u : \Omega \to \mathbb{R}^2$ satisfying the nonlinear differential inclusion

$$\begin{cases} Du \in E, & \text{a.e. in } \Omega \\ u = \varphi, & \text{on } \partial\Omega \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^2$ and E is a compact subset of $\mathbb{R}^{2\times 2}$, which is isotropic, that is to say, invariant under orthogonal transformations. Our result relies on an abstract existence theorem due to Müller and Šverák which requires the set E to satisfy a certain in-approximation property.

Keywords. Differential inclusion, isotropic set, singular values, rank one convexity, quasiconvexity, polyconvexity.

1 Introduction

A microstructure is a structure on a scale between the macroscopic and the atomic ones. Microstructures are abundant in nature, for example, they are present in molecular tissues or in biomaterials. Crystals such as igneous rocks or metal alloys (for example nickel-aluminium, zinc-lead) also develop microstructures. The microstructure of a material can strongly influence physical properties such as strength, toughness, ductility, hardness and corrosion resistance. This influence can vary as a function of the temperature of the material.

In the last twenty years successful models for studying the behaviour of crystal lattices undergoing solid-solid phase transitions have been studied. In such models it is assumed that the elements of crystal lattices have certain preferable affine deformations; this is true for example for martensite or for quartz crystals (see [1, 15]).

Denoting by E the set of matrices corresponding to the gradient of these deformations, the physical models motivate the mathematical question of the existence of solutions to Dirichlet problems related to systems of differential inclusions such as $Du \in E$ a.e. in Ω, $u = \varphi$ on $\partial\Omega$, where Ω is a domain of $\mathbb{R}^n$ and $E \subset \mathbb{R}^{n\times n}$ is a compact set.

Two abstract theories to establish the existence of solutions of general differential inclusion problems are due to Dacorogna and Marcellini (see [8, 6]), whose result is based on Baire's category theorem, and Müller and Šverák [16, 17], who use ideas of convex integration by Gromov [12]. In these two theories certain convex hulls of the set E play an important role. We say that a set $E \subseteq \mathbb{R}^{n\times n}$ is rank one convex if for every $\xi, \eta \in E$ such that $\text{rank}(\xi - \eta) = 1$ and for every $t \in [0, 1]$ then

$$t\xi + (1 - t)\eta \in E.$$

In the spirit of the usual definition of the convex hull, the rank one convex hull of a set E can be defined as the smallest rank one convex set that contains E. However, in order to solve differential inclusion problems, several authors, namely Müller and Šverák [16, 17], consider the following alternative notion of the rank one convex hull of a compact set $E \subset \mathbb{R}^{N\times n}$, denoted by E^{rc}:

$$E^{rc} = \big\{\xi \in \mathbb{R}^{N\times n} : f(\xi) \le 0, \text{ for every rank one convex function } f \in \mathcal{F}^E\big\}$$

where

$$\mathcal{F}^E = \big\{f : \mathbb{R}^{N\times n} \to \mathbb{R} : f\lfloor_E \le 0\big\}.$$

In each of the aforementioned theories, provided certain approximation properties hold, if the gradient of the boundary datum φ belongs to the interior of the appropriate convex hull of E, then there exists a solution $u \in \varphi + W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ to $Du \in E$ a.e. in Ω.

Using these abstract theorems various interesting problems related to the existence of microstructures have been solved, such as the two-well problem, where $E = \mathcal{SO}(2)A \cup \mathcal{SO}(2)B$, where A and B are two fixed $\mathbb{R}^{2\times 2}$ matrices and $\mathcal{SO}(2)$ stands for the special orthogonal group (see [7, 8, 11, 15, 16]).

In this article we study the case where the set E is an arbitrary $\mathbb{R}^{2\times 2}$ isotropic set, that is, invariant under orthogonal transformations. More precisely, we assume that E is a compact subset of $\mathbb{R}^{2\times 2}$ such that $RES \subseteq E$ for every R, S in the orthogonal group $\mathcal{O}(2)$. Let Ω be an open bounded subset of $\mathbb{R}^2$. We investigate the existence

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