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SOME NEW RESULTS ON SENSITIVITY ANALYSIS OF HYBRID SYSTEMS

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Abstract. An overview of new results on the sensitivity analysis of two categories of hybrid system is presented. Firstly, a theory is presented for local, first-order sensitivity analysis of limit-cycle oscillating hybrid systems, which are dynamical systems exhibiting both continuous-state and discrete-state dynamics whose state trajectories are closed, isolated and time-periodic. A method is described for decomposition of the parametric sensitivities into three parts, characterizing the influence of parameter changes on period, state variable amplitudes, and relative phases, respectively. The computation of parametric sensitivities of period, amplitudes, and phases is also described. It is shown that, in general, it is incorrect to set the initial conditions for parametric sensitivities of state variables to zero. Secondly, using results from nonsmooth analysis, sufficient conditions are presented for the existence of the forward and adjoint sensitivities of parametric ordinary differential equations with locally Lipschitz continuous vector fields. Unique aspects of these results are demonstrated via examples.

Keywords. hybrid system, sensitivity analysis, limit cycle, amplitude sensitivity, phase sensitivity, boundary-value problem, adjoint sensitivity, forward sensitivity, nonlinear ODEs, nonsmooth analysis.

AMS subject classification. 37C27, 49Q12, 49J52, 65L99

1 Introduction

Hybrid limit-cycle oscillators (HLCOs) are hybrid discrete/continuous dynamical systems whose state trajectories are time-periodic, closed and isolated. Examples of HLCOs include models of the cell cycle [4, 5] in which a cell's mass drops instantaneously upon mitosis. Other examples include models of walking and hopping robots [10, 13, 15], in which state variables change discontinuously whenever a robot's foot touches the ground.

Proofs of the properties of continuous limit-cycle oscillators [14] and the development of sensitivity analysis of limit-cycle oscillators [16, 20] do not extend directly to HLCOs. The results summarized in Section 2 bridge and extend previous work on the sensitivity analysis of continuous-state limit-cycle oscillators [16, 20] and hybrid systems [2, 9].

The focus of Section 3 is the existence of the derivative of the mapping $\boldsymbol{\eta} \mapsto \mathbf{x}(t_f, \boldsymbol{\eta})$ at $\mathbf{p} \in \mathcal{P}$, where $\mathbf{x}\colon [t_0, t_f] \times \mathcal{P} \to \mathcal{X}$ is the solution of the initial value problem:

$$\dot{\mathbf{x}}(t, \mathbf{p}) = \mathbf{f}(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p})),\ \forall t \in (t_0, t_f],$$
$$\mathbf{x}(t_0, \mathbf{p}) = \mathbf{f}_0(\mathbf{p}),\ \forall \mathbf{p} \in \mathcal{P} \subset \mathbb{R}^{n_p},$$

where $\mathbf{f}\colon \mathcal{T} \times \mathcal{P} \times \mathcal{X} \to \mathbb{R}^{n_x}$ and $\mathbf{f}_0\colon \mathcal{P} \to \tilde{\mathcal{X}}$ are locally Lipschitz continuous functions, $\mathcal{T}$ is an open subset of $\mathbb{R}$ such that $[t_0, t_f] \subset \mathcal{T}$, $\mathcal{X}$ is an open subset of $\mathbb{R}^{n_x}$, $\tilde{\mathcal{X}}$ is an open subset of $\mathcal{X}$, $\mathcal{P}$ is an open subset of $\mathbb{R}^{n_p}$, n_p and n_x are positive finite integers. If the derivative of the mapping $\boldsymbol{\eta} \mapsto \int_{t_0}^{t_f} g(t, \boldsymbol{\eta}, \mathbf{x}(t, \boldsymbol{\eta}))\mathrm{d}t$ is required, the integration of the adjoint sensitivity initial value problem can be a computationally more efficient way of obtaining this derivative. The adjoint, $\boldsymbol{\lambda}\colon [t_0, t_f] \to \mathbb{R}^{n_x}$, is an absolutely continuous function, and the desired derivative is ultimately computed as an integral of the form $\int_{t_0}^{t_f} \mathbf{h}(t, \boldsymbol{\eta}, \mathbf{x}(t, \boldsymbol{\eta}), \boldsymbol{\lambda}(t))\mathrm{d}t$. In the case that $\mathbf{f}$ is continuously differentiable, the development of forward and adjoint sensitivities can be found in [3, 7].

2 Hybrid Limit-Cycle Oscillators

The contents of this section are a summary of the formulation and results developed in [12], where detailed proofs may be found.

2.1 Definition and dynamic behavior

The hybrid oscillators considered in this section are described by the following definition:

Definition 2.1. An *oscillating hybrid system* is a nondimensionalized 11-tuple

$$\mathcal{H} = (\mathcal{M}, \mathbf{p}, T, \mathcal{E}, \mathcal{T}_{\mathcal{M}}, \mathbf{x}, \mathbf{x}_0, \mathbf{F}, \mathcal{L}, \boldsymbol{\sigma}, \boldsymbol{\Theta}),$$

with the elements of $\mathcal{H}$ defined as follows:

1. $\mathcal{M} = \{1, 2, \ldots, n_m\}$, for some $n_m \in \mathbb{N}$,
2. $\mathbf{p} \in P$, where P is an open subset of $\mathbb{R}^{n_p}$ for some $n_p \in \mathbb{N}$,

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