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LOCAL SMOOTHING EFFECT AND EXISTENCE FOR THE ONE-PHASE HELE-SHAW PROBLEM WITH SURFACE TENSION

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Abstract. We study an initial value problem for the one phase Hele-Shaw problem with surface tension and with injection (or suction). We establish local well-posedness for the initial value problem in Sobolev space. Furthermore, we obtain that, on average in time, the solution gains 3/2 derivative of smoothness in spatial variable compared to the initial data. This smoothing effect is a direct consequence of the fact that the Hele-Shaw problem with surface tension is uniformly parabolic.

Key words. Unsteady Hele-Shaw problem, Sobolev space, initial value problem, well posedness, smoothing effect.

AMS subject classifications. 35R35, 76N05, 80A22.

1 Introduction

We consider the time-evolving displacement of a viscous fluid by another fluid of less viscosity in a Hele-Shaw cell. As documented on Howison's website [28],these free surface flows have been the subject of numerous works in the last 50 years since the early work of Polubarinova-Kochina [36], Galin [17] and Saffman-Taylor [42]. Some recent reviews and books are Cummings et al [11], Tanveer [45] and Gustafsson and Vasilev [22].

We study the one phase Hele-Shaw problem. Let $\Omega(t)$ in the (x, y) plane be the fluid domain with free boundary $\partial\Omega$; this may be a finite simply-connected blob, the exterior of a finite or infinite inviscid bubble. The motion is driven by singularities such as sources, sinks or multipoles within the fluid region (and possibly at infinity). The fluid velocity averaged across the gap is $u = -\nabla p(x, y, t)$ where p is the pressure. For an incompressible fluid

$$\Delta p = \gamma\delta(z_0) \text{ in } \Omega(t) \qquad (1.1)$$

where $\delta(z_0)$ is the Dirac function and γ is the rate of strength of injection or suction at $z_0 = (x_0, y_0) \in \Omega(t)$ which can be assumed to be 0 or ∞. Together with the dynamic boundary condition

$$p = 0 \text{ on } \partial\Omega(t) \qquad (1.2)$$

or

$$p = \tau\kappa \text{ on } \partial\Omega(t) \qquad (1.3)$$

where τ is a dimensionless surface tension coefficient and κ is the curvature of the free boundary and the kinematic boundary condition

$$-\frac{\partial p}{\partial n} = V_n \text{ on } \partial\Omega(t), \qquad (1.4)$$

where $\frac{\partial}{\partial n}$ denotes the derivative in the direction of the outward normal n to $\partial\Omega$, and V_n is the velocity of the $\partial\Omega(t)$ in the direction of n.

For zero surface tension problem, explicit solutions were obtained by Saffman-Taylor [42],Richardson [38], Howison [23],Cummings et al [11], Bensimon et al [4]. Gustafson [21] and DiBenedetto and Friedman [13] have proven existence result based on variational formulation of the problem (see also Andreucci et al [2]); Gustafson [20] and Reissig [40, 41] proved the existence of analytic solution if the initial data is analytic. Kim [31, 32] obtained existence results of viscosity solution.

For nonzero surface tension case, Duchon & Robert [14] have proven the existence for short time. Constantin and Pugh [10] proved the global existence for small analytic initial data and ; Tanveer [44] analyzed the curvature-induced complex singularities based on asymptotic analysis. Escher and Simonett [16] discussed solutions in Holder space while Prokert [37] obtained existence in Sobolev spaces. Recently Gunther and Prockert [18, 19] obtained existence results for a similar problem with variable surface energy; Bazaliy and Friedman [3] have obtained the existence in Holder space for a similar problem in a half plane.

For selection of steady fingers and bubbles in Hele-Shaw cell, we refer to [7, 45, 47, 51, 52].

For two phase Hele-Shaw problem, we refer to Howison [27], Chen [8], Siegel et al [43], Ambrose [1] and Entov and Entigov [15] and Ye and Tanveer [53].

The unsteady Hele-Shaw problems with suction are ill posed in Hadamard sense, the formation and nature of singularity for this problem is discussed

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