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FRACTIONAL RUNGE-KUTTA METHODS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. Based on high order approximation of L-stable Runge-Kutta methods for the Riemann-Liouville fractional derivatives, several classes of high order fractional Runge-Kutta methods for solving nonlinear fractional differential equation are constructed. Consistency, convergence and stability analysis of the numerical methods are given. Numerical experiments show that the proposed methods are efficient for solving nonlinear fractional differential equation.

Keywords. Nonlinear fractional differential equation; fractional Runge-Kutta methods; consistency; convergence; stability.

AMS subject classifications. 26A33, 65L05, 65L06, 65L20.

1 Introduction

There has been increasing interest in the description of dynamical system by means of equations involving fractional derivatives and integrals over the last decades(see [16,17,18]), then Fractional Differential Equations (FDEs) naturally arise. FDEs can arise in which both the temporal derivative and spatial derivative operators are fractional, but in this paper, we only consider fractional derivative operators with respect to time.

For numerical computation of the Riemann-Liouville fractional integrals and fractional derivatives, Lubich(see [1]) extended BDF methods to obtain the approximation of fractional integrals and fractional derivatives. Lubich and Ostermann(see [4]) derived the approximation of convolution integrals and integral equations based on Runge-kutta methods, and given the result of the error analysis. In 2004, Lubich(see [5]) given further the approximate the convolution with singular kernel. These work provide with the base for the research of high efficient numerical algorithms of FDEs. Much work has been done on developing numerical methods for FDEs. Diethelm et al.(see[6,7,8,9,10,11]) have done a lot of excellent works for FDEs. They considered the use of the trapezoidal method, predictor-corrector method, finite difference method and extrapolation. Ford et al.(see[12,13]) compare the methods proposed by several authors and present a new approach to reducing computational cost that keeps the error under control. Lin et al.(see [14]) constructed the fractional BDF methods for solving nonlinear FDEs. Cao et al.(see [15]) presented the fractional Radau IIA methods for solving nonlinear FDEs.

In fact, the main characteristic of FDEs is that any numerical approximation of a fractional derivative needs past information all the way back to the initial value. Thus a FDEs is said to have memory. This makes the computation very expensive. So efficient numerical algorithms for solving FDEs are imperative. The objective of this paper is to achieve the efficient numerical methods with good stability properties and high accuracy.

The outline of the paper is as follows. In section 2, the numerical methods for nonlinear FDE are proposed. Then, in section 3 , 4 and 5, consistency, convergence and stability analysis of fractional Runge-Kutta methods are obtained. Section 6 compare fixed stepsize implementation for different numerical methods on a selection of test problem. The advantages of the numerical methods for nonlinear FDE are that they can combine good stability with high accuracy.

Consider the following nonlinear fractional differential equation(NFDE)

$$\begin{cases} {}_0^C D_t^\alpha y(t) = f(t, y(t)), & 0 \le l-1 < \alpha < l, 0 \le t \le T, \\ y^{(i)}(0) = y_0^{(i)}, & i = 0, 1, \cdots, l-1, \end{cases} \tag{1}$$

where ${}_0^C D_t^\alpha y(t)$ denotes the Caputo fractional derivative of order α of the function $y(t)$, defined by

$${}_0^C D_t^\alpha y(t) = \frac{1}{\Gamma(l-\alpha)} \int_0^t \frac{y^{(l)}(\tau)d\tau}{(t-\tau)^{\alpha-l+1}},$$

$f : [0,T] \times R \to R$ is a given continuous mapping and satisfies the Lipschitz condition with respect to $y(t)$

$$| f(t, y) - f(t, z) | \le L | y - z |, \quad y, z \in R. \tag{2}$$

Throughout this paper we assume that the problem (1) has a unique exact solution $y(t)$.

Using the relationship between Caputo fractional derivative and Riemann-Liouville fractional derivative, we have

$${}_0 D_t^\alpha y(t) = f(t, y(t)) + \sum_{k=0}^{l-1} \frac{t^{k-\alpha} y^{(k)}(0)}{\Gamma(k-\alpha+1)}, \tag{3}$$

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