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CAUCHY BIORTHOGONAL POLYNOMIALS AND DISCRETE INTEGRABLE SYSTEMS

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Abstract. Discrete spectral transformations of the Cauchy biorthogonal polynomials are constructed. From the compatibility conditions of the spectral transformations, two discrete integrable systems are derived. Two class of their solutions, associated with the Cauchy biorthogonal polynomials and with the quasi-orthogonal polynomials are explicitly given.

Keywords. Cauchy biorthogonal polynomials, quasi-orthogonal polynomials, discrete integrable systems, spectral transformation, Lax pair.

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1 Introduction

It is well-known that the integrable systems have a rich connection to many fields including dynamical systems, quantum physics and special functions. There has been growing interest in the studies of the relationship between the integrable systems and the orthogonal functions [11]. The orthogonal polynomials appear both in continuous time and discrete time as wave functions of the Lax pair of the Toda lattice . From the point of view of the theory of the orthogonal polynomials, the Lax pair can be regarded as the spectral transformation of sequence of the orthogonal polynomials [10]. It is also shown that new class of discrete integrable systems can be derived from the discrete transformation of biorthogonal functions, and some class of their solutions are analyzed in detail [9, 12].

Recently, Bertola et al. proposed a new biorthogonal polynomials, called Cauchy biorthogonal polynomials (CBOPs), for the Hermite-Padé type approximation problems associated with the inverse problems of the Degasperis-Procesi equation [2]. Some type of random matrix models has been discussed by using the prosperous property of CBOPs [1]. One of the specific features of CBOPs is the four-term recurrence relation, which plays an important role in constructing the Christoffel-Darboux like identities used in formulating Riemann-Hilbert problems and calculating the distribution of the random matrix model [1, 2]. On the other hand, quasi-orthogonal polynomials (Q-OPs), proposed in the generalization of the moment problem of the orthogonal polynomials in 1923 [7], hold the same recurrence relation [5]. As for Q-OPs, the corresponding discrete integrable system also has not been considered.

In application of the CBOPs to the numerical algorithms, one of the prospective strategies is to derive the discrete integrable systems associated with CBOPs, since many number of discrete integrable systems are related to the eigen value problems [8, 13]. The main purpose of this paper is to clarify the transformation theory of the CBOPs in order to derive discrete integrable systems associated with CBOPs and with Q-OPs.

The contents of this paper is as follows. In section 2, we recall on CBOPs briefly. Especially we show that CBOPs hold some type of four-term recurrence relation. In section 3, we construct the discrete transformation of CBOPs and derive the discrete integrable systems associated with CBOPs. Additionally we also present the solution of these systems explicitly. In section 4, we discuss the case of Q-OPs in a similar manner to CBOPs. Finally we conclude this paper.

2 Cauchy biorthogonal polynomials

In this section , we give a brief review of CBOPs [1]. First we give a definition of CBOPs.

Definition 2.1. The pair of the sequences of the polynomials $(\{p_m(x)\}_{m=0}^{\infty}, \{q_n(y)\}_{n=0}^{\infty})$ are called CBOPs with respect to the bilinear 2-form $\langle\cdot|\cdot\rangle$ if they satisfy

$$\langle p_m(x)|q_m(y)\rangle = h_n\delta_{mn} \quad (\exists h_n \neq 0), \tag{1}$$
$$\deg(p_n(x)) = \deg(q_n(y)) = n, \tag{2}$$

where

$$\langle f(x)|g(y)\rangle = \iint \frac{f(x)g(y)}{x+y} d\rho_1(x)d\rho_2(y). \tag{3}$$

Remark 2.1. We assume that the domain of integration and the Stieltjes functions $d\rho_1, d\rho_2$ of $\langle\cdot|\cdot\rangle$ satisfy these two conditions:

(i) The moment $\langle x^i|y^j\rangle$ is finite for all $i.j \in \mathbb{Z}_{\geq 0}$.

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