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STRUCTURE RESULTS FOR HIGHER ORDER SYMMETRY ALGEBRAS OF 2D CLASSICAL SUPERINTEGRABLE SYSTEMS

E. G. Kalnins and W. Miller Jr. *†‡

Abstract. Recently the authors and J.M. Kress presented a special function recurrence relation method to prove quantum superintegrability of an integrable 2D system. The procedure included explicit constructions of higher order symmetries and the structure relations for the closed algebra generated by these symmetries. We applied the method to 5 families of systems, each depending on a rational parameter k, including most notably the caged anisotropic oscillator, the Tremblay, Turbiner and Winternitz system and a deformed Kepler-Coulomb system. Here we work out the analogs of these constructions for all of the associated classical Hamiltonian systems, as well as for a family including the generic potential on the 2-sphere. We do not have a proof in every case that the generating symmetries are of lowest possible order, but we believe this to be so via an extension of our method.
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1 Introduction

Recently Tremblay, Turbiner and Winternitz [1, 2] studied a family of quantum and classical mechanical systems in two dimensions with classical Hamiltonian

$$\mathcal{H} = p_r^2 + \frac{1}{r^2}p_\theta^2 - \omega^2 r^2 + \frac{\alpha}{r^2\cos^2 k\theta} + \frac{\beta}{r^2\sin^2 k\theta}. \quad (1)$$

We call this the TTW system. They conjectured and gave strong evidence that for k a rational number this system is superintegrable, as is the corresponding quantum analog

$$H = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 - \omega^2 r^2 + \frac{\alpha}{r^2\cos^2 k\theta} + \frac{\beta}{r^2\sin^2 k\theta}. \quad (2)$$

In papers [3, 4] the authors and collaborators proved quantum and classical superintegrability for these systems, the first proofs that covered all rational k. Related results are [5, 6, 7]. This system has sparked a great deal of interest because it includes several special cases of definite physical interest, it provides many examples of superintegrability where the generating constants of the motion are of very high order as polynomials in the momenta or as differential operators in the quantum case, and because it gave a hint as to how many other such families of classical and quantum systems could be constructed. The previous classical proofs of superintegrability for these systems have usually not determined the structure of the symmetry algebras, or even verified closure at finite order under commutation. For 2D quantum systems, however, the authors and J. M. Kress have recently introduced a method for verifying superintegrability based on recurrence relations for hypergeometric functions that enables one to compute the associated structure relations for the symmetry algebra with relative ease. In [7] structure relations were obtained for 5 families of superintegrable systems, each indexed by a rational parameter k, and including the TTW family. Here we give a simplified exposition of some of these results.

Each of these quantum systems in 2D has an analogous classical counterpart that can be shown to be classically superintegrable. Natural questions are

- What is the structure of the classical symmetry algebra for each of these systems? Does it close at finite order?
- How are the structures of the classical and quantum symmetry algebras, and the symmetries themselves, related?

We will present new results in this paper that provide at least partial answers to these questions for each of the systems studied in [7], as well as for a family that contains the 2D system on the 2-sphere with generic potential.

2 Brief resume of definitions

Suppose we have a Hamiltonian system on a 2D local Riemannian manifold (real or complex) such that the Hamilton-Jacobi equation admits additive separation in some orthogonal coordinate system. Then the separable coordinates x_1, x_2 can always be chosen such that the

*E. G. Kalnins is with the Department of of Mathematics, University of Waikato, Hamilton, New Zealand. E-mail: math0236@math.waikato.ac.nz
†W. Miller Jr. is with the School of Mathematics, University of Minnesota. Minneapolis, Minnesota, USA. E-mail: miller@ima.umn.edu
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