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ATTRACTIVITY OF SOLUTIONS IN SECOND ORDER DIFFERENTIAL EQUATIONS WITH UNBOUNDED DAMPING

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Abstract. Sufficient conditions for global attractivity of solutions of the damped oscillator equation

$$x'' + h(t)x' + x = F(t)$$

are obtained for unbounded damping $h(t)$. The method used does not depend on linearity and is applied to obtain a similar result to the perturbed nonlinear equation of the form

$$x'' + h(t, x, x')x' + g(x) = q(t, x, x').$$

The aim of this paper is to give integral conditions on the damping term h guaranteeing attractivity of solutions, yet permitting this coefficient to grow to infinite with time and keeping its lower bound as small as possible. The results extend several early works on the subject.

Keywords. Asymptotic stability, Global attractivity, Unbounded damping

1 Introduction

The differential equation

$$x'' + h(t)x' + x = 0 \tag{1}$$

where $h : [0, \infty) \to [0, \infty)$ is a piecewise continuous function, describes the motion of an oscillator under the action of viscous friction with time-varying damping coefficient $h(t)$. Equation (1) and its perturbation

$$x'' + h(t)x' + x = F(t) \tag{2}$$

also serve as basic model equations in mechanical and electrical vibrations, traveling waves, and structural stability. $F : [0, \infty) \to [0, \infty)$ is again piecewise continuous.

Numerous investigators have searched for conditions on the damping function $h(t)$ to ensure the asymptotic stability of the equilibrium state $x = x' = 0$, which means that $x(t) \to 0, x'(t) \to 0$ as $t \to \infty$ for every solution x of (1) (see [1, 2, 3, 5, 6, 7, 8, 9, 11, 13, 15, 16, 17]). Investigators have also become increasingly interested in properties and techniques that are valid for wider classes of problems when (2) is written in form of delay or damped wave equations (see [3, 4, 12, 17, 18, 19]). It is well-known that if the damping function is restricted to $0 < \alpha \le h(t) \le \beta$ for some constants α, β, then the equilibrium state of (1) is asymptotically stable (see [9]). On the other hand, it is also known that if the damping function h(t) is not bounded above (for example $h(t) = 2+e^t$), the equilibrium state is not necessarily asymptotic stable and solutions may exist such that $x(t) \to c \neq 0$ as $t \to \infty$ (see [6]). Moreover, $\int_0^\infty h(t)dt = \infty$ is a necessary condition for the asymptotic stability of the equilibrium state of (1) (see [15]). This means that the damping function cannot be too "small" or too "big" in some sense. Significant progress has been made in the search for the right $h(t)$. For example, the upper bound for $h(t)$ has been improved to

$$\int_0^t h(s)ds \le Bt^2 \tag{3}$$

for all $t \ge 1$ and a constant B. A necessary and sufficient condition in terms of integral of h is also known (see [15]), but not easy to check. A simplified version of this condition can be found in [8]. Under condition (3) and $h(t) \ge \alpha > 0$, Artstein and Infante ([1]) showed that every solution x of (1) satisfies

$$(x(t), x'(t)) \to (0, 0) \ \text{ as } \ t \to \infty. \tag{4}$$

In Section 2, we show that if h satisfies (3) and h is integrally positive, then every solution x of (2) satisfies (4) whenever $F \in L^1[0, \infty)$. We also provide an example showing that condition (3) is sharp in sense that if Bt^2 is replaced by $Bt^{2+\varepsilon}$ in (3) for any $0 < \varepsilon < 1$, then the global attractivity of solutions in (2) is no longer guaranteed. The result extends the theorem mentioned above and will be generalized to nonlinear equations in Sections 3.

The technique of qualitative analysis used here is inspired by the works in ([1, 3, 7]) and can as well be generalized to a wider class of equations such as

$$x'' + h(t, x, x')x' + g(x) = q(t, x, x') \tag{5}$$

where $xg(x) > 0$ for $x \neq 0$, $h(t, x, y) \ge 0$, and

$$|q(t, x, y)| \le \tilde{q}(t)(1 + |x| + |y|) \text{ with } \tilde{q} \in L^1[0, \infty).$$

This will be done in Section 3 providing some new results in attractivity and perturbation for nonlinear equations

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