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A UNIFIED DOUBLE INTEGRAL ASSOCIATED WITH WHITTAKER FUNCTIONS

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Abstract. The aim of the research paper is to evaluate integral formulas involving the Whittaker function of first kind $M_{k,\mu}(z)$, which are expressed in terms of Kampé de Fériet functions. Some interesting special cases of our main results involving exponential function, Modified Bessel function, Sine hyperbolic function, Laguerre polynomials and Hermite polynomials are expressed in terms of Kampé de Fériet functions.

Keywords. Whittaker function, Kampé de Fériet functions, generalized hypergeometric functions and Integrals.

1 Introduction

A large number of integral formulas involving a variety of special functions have been developed by a number of authors (see, [4], [5], [8] and [1] and [9]). Many integral formulas involving a product of Whittaker function have been developed and play an important role in several physical problems. In particular, Whittaker function are associated with wide range of problems in diverse area of mathematical physics. These connections of Whittaker functions with various other research areas have led many researchers to the field of special functions. Motivated by the work of Choi and Agarwal [6] and Khan and Ghayasuddin [9], Here, the authors are to present unified double integral formulas involving Whittaker function of the first kind $M_{k,\mu}(z)$, which are expressed in terms of the Kampé de Fériet functions.

The Whittaker functions $M_{k,\mu}(z)$ and $W_{k,\mu}(z)$ was introduced by Whittaker [13] (see also Whittaker and Watson [14]) in terms of confluent hypergeometric function ${}_1F_1$ (or Kummer's functions):

$$M_{k,\mu}(z) = z^{\mu+\frac{1}{2}} \; e^{-z/2} \; {}_1F_1\left(\frac{1}{2}+\mu-k \; ; 2\mu+1 \; ; z\right). \tag{1.1}$$

and

$$W_{k,\mu}(z) = z^{\mu+\frac{1}{2}} \; e^{-z/2} \; U\left(\frac{1}{2}+\mu-k \; ; 2\mu+1 \; ; z\right). \tag{1.2}$$

However the confluent hypergeometric function disappears when 2μ is an integer, so whittaker functions are often defined instead. The whittaker functions are related to the parabolic cylinder functions.

When $\mid arg(z) \mid < \frac{3\Pi}{2}$ and 2μ is not an integer,

$$W_{k,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-k)}\, M_{k,\mu(z)} + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-k)} M_{k,-\mu}(z), \tag{1.3}$$

When $\mid arg(-z) \mid < \frac{3\Pi}{2}$ and 2μ is not an integer,

$$\begin{aligned} W_{-k,\mu}(z) = {} & \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-k)}\, M_{-k,\mu(-z)} \\ & + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu+k)} M_{-k,-\mu}(-z). \end{aligned} \tag{1.4}$$

In 1921, Appell's four hypergeometric functions [2] F_1, F_2, F_3, F_4 and seven his confluent forms $\phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \Xi_1, \Xi_2$ were unified and generalized by Kampé de Fériet [3]. We recall here the definition of a more general double hypergeometric function of Kampé de Fériet in a slightly modified notation of Shivastava and Panda(see [12]):

$$F^{A:\;B;\;D}_{E:\;G;\;H}\left[\begin{array}{llll} (a_A): & (b_B); & (d_D); & \\ (e_E): & (g_G); & (h_H); & x, y \end{array}\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{[(a_A)]_{m+n}\;[(b_B)]_m\;[(d_D)]_n}{[(e_E)]_{m+n}\;(g_G)_m\;(h_H)_n}\;\frac{x^m}{m!}\;\frac{y^n}{n!}, \tag{1.5}$$

where, for convergence,

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