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THE PHASE SPACE OF HIGHER-DERIVATIVE THEORIES

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Abstract. The phase space of a higher-derivative gravity theory is demonstrated to be equivalent to the space of solutions to the field equations only when quadratic functions of first derivatives occur of the variables occur in the action. The projection from phase space to the space of solutions for higher-derivative theories removes the symplectic invariance and the existence of flat connections may not be derived from field equations of a higher-dimensional Chern-Simons or self-dual Yang-Mills theory related to string theory, except for Lovelock actions which have second-order field equations.

Keywords: phase space, higher-derivative equations, symplectic invariance

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1 Introduction

Since the phase space of the three-dimensional gravitational action is equivalent to the space of solutions to the field equations, it is possible to identify the formulae for the triad of orthonormal basis vectors with flat connectons of a Chern-Simons theory with an $ISO(2,1)$ invariance [1]. The equivalence is valid for all Lagrangians consisting of terms quadratic in the derivatives of the coordinates.

With the resolution of problems with infinities in the perturbation series and singularities in the space-time through string theory, the metric of the embedding space-time must satisfy equations necessary for conformal invariance of the two-dimensional sigma model which may be derived from an effective action with higher-order curvature combinations. The heterotic string effective action may be reduced at second order in this series to a quadratic gravity theory [2].

The theory with higher derivatives of the scale factor that arise in the integral in a Friedman-Robertson-Walker space-time may be quantized in the Ostrogradski formalism. The phase space is expanded to include more conjugate variables. The action is linear only in the second derivative of the scale factor, and it differs from an integral with only first derivatives by a boundary term.

However, more general actions could yield quadratic functions of higher derivatives of the metric and the scale factor in this minisuperspace. Then the space of solutions to the field equations must be mapped to a subspace of the phase space spanned by the fields and conjugate momenta. The equivalence is not preserved if the the dimensions of the spaces differ or the Jacobian matrix does not have the maximum rank at generic values of the coordinates and the derivatives. Given the absence of symplectic invariance, these gravity theories could not be recast in the form of Chern-Simons or Yang-Mills theories representing a string effective action. This result confirms the conclusion regarding field and metric redefinitions such that the equations are second-order.

2 The Phase Space of the Quadratic Gravity Model

The quadratic gravity model derived from heterotic string theory in the minisuperspace of Friedman-Robertson-Walker metrics [3] equals

$$I = \int dt \left[(6a^2\ddot{a} + 6a\dot{a}^2 + \frac{1}{2}a^3\dot{\Phi}^2 + 6\frac{e^{-\Phi}}{g_4^2}\ddot{a}(\dot{a}^2+K) \right]. \quad (1)$$

There is a second-order derivative of the scale factor in the action, although it occurs only linearly in the integral. The momentum space variables conjugate to $(a, \dot{a}, \Phi)$ in the Ostrogradski formalism are

$$P_a = \frac{\partial L}{\partial \dot{a}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{a}}\right) = 6\frac{e^{-\Phi}}{g_4^2}\dot{\Phi}(\dot{a}^2+K) \quad (2)$$

$$P_{\dot{a}} = \frac{\partial L}{\partial \ddot{a}} = 6\left(a^2 + \frac{e^{-\Phi}}{g_4^2}(\dot{a}^2+K)\right) \quad (3)$$

$$P_\Phi = \frac{\partial L}{\partial \dot{\Phi}} = a^3\dot{\Phi}. \quad (4)$$

$$(5)$$

The phase space $\{a, \dot{a}, \Phi, P_a, P_{\dot{a}}, P_\Phi\}$ has a volume element of the form

$$da \wedge dP_a \wedge d\dot{a} \wedge dP_{\dot{a}} \wedge d\Phi \wedge dP_\Phi \quad (6)$$

which is invariant under symplectic transformations

$$\begin{pmatrix} a \\ P_a \\ \dot{a} \\ P_{\dot{a}} \\ \Phi \\ P_{\dot{\Phi}} \end{pmatrix} \to \begin{pmatrix} a \\ P_a \\ \dot{a} \\ P_{\dot{a}} \\ \Phi \\ P_{\dot{\Phi}} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \quad (7)$$

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