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# BIFURCATION ANALYSIS FOR A CLASS OF CONTRACTIVE PIECEWISE SMOOTH SYSTEMS

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**Abstract.** In this paper, the influence of a class of quadratic perturbations on contractive piecewise smooth systems is investigated. Based on Lyapounov-Schmidt approach, the relied bifurcation analysis shows that three scenarios are possible: the contractive behavior is preserved, may bifurcate into two different contractive behaviors or contraction property is completely lost.
**Keywords:** Contraction theory; piecewise smooth systems; bifurcations; Lyapounov-Schmidt method.
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## 1 Introduction

The aim of this paper is to analyze the impact of some perturbations on a class of contractive piecewise smooth systems (noted p.w.s systems). In the gamut of dynamic systems with discontinuities, the most general class corresponds to hybrid systems which are compound of continuous and discrete dynamics. The fundamental concepts on hybrid systems are not yet developed as for smooth systems. Some important textbooks available are given in [57] and qualitative results are proposed in [39, 44] for some classes of hybrid systems. However, an important class of hybrid systems is those of p.w.s systems, they are defined by an ordinary differential equation in each region, are continuous across the border but their derivatives are discontinuous. Those systems modelize a vast range of physical systems involving non smooth behavior [36], [11], [25]. Authors in [7] proposed interesting treatments and applications in this context. Nowadays, it has been recognized that those systems exhibit many interesting phenomena because of the complex structure of the state space composed of different vector fields, see for example the overview of some aspects of chaotic behavior for those systems in [13]. For journal and conferences papers, the reader can also refer the recent overview articles [22] for numerous references therein.

In this paper, we focus our attention on p.w.s systems defined as:

$$\dot{x} = f_i(t, x(t)) \quad \text{for } t \in [t_{i-1}, t_i[\,,\ i = 1, ..., p \tag{1}$$

where $p \in N^*$, $f_i$ are functions defined on open disjoint sets $[t_{i-1}, t_i[ \times D_i$ , and $D_i := \{x(t)\epsilon R^n : t \in [t_{i-1}, t_i[\}$ .

Regions $D_i$ and $D_j$ are disjoints open sets of $R^n$, non empty and separated by manifolds $\Sigma_{ij} := \bar{D}_i, \cap \bar{D}_j$ where $\bar{D}_i$ is the closure of $D_i$, $\Sigma_{ij}$ are called switching manifolds and intersect transversely the domain $\bar{D} := \cup \bar{D}_i$.

The system (1) is defined on the domain $W := I \times \bar{D}$, where $I := [t_0, T]$, $T \in R^*_+$ being fixed.

As we focus particularly on contraction property of (1), some recalls on contraction theory are first presented whereas the bifurcation problem relied to this property under some perturbations of the system dynamics will be investigated based on Lyapounov-Schmidt approach and illustrated by some simulation results.

Many authors studied qualitative aspects of dynamic systems in incremental stability context (called also relative stability) that is inspired by fluid mechanics (i.e. a spacial and global vision of flow trajectories, $\dot{x}$ being the velocity vector at position $x$ and time $t$). Roughly speaking, this approach concerns the attractivity or repulsivity study between trajectories rather than with respect to some predefined attractor. In this frame, it is important to note that three classes of incremental stability methods are known: contraction, incremental gains techniques and $\delta$-stability. In this work, we are concerned by contraction theory defined in functional analysis frame witch is the appropriate context for our proposed study.

In fact, contraction process is a stability theory where stability is defined incrementally between two arbitrary trajectories. It means that at any time, any two initial points of two very close trajectories (separated by an infinitesimal distance of the considered system) will go together in a same direction, the whole flow will "schrink" to a same moving direction. This is possible under existence of a contraction metric for witch the related Jacobian matrix measure of the system is uniformly definite negative over some $K-$reachable set [1] of the phase space. This ensures that a suitably defined distance between nearby trajectories is always decreasing and thus trajectories converge incrementally exponentially and globally in this set. It's different from the concept of Lyapounov stability since particularly existence of an equilibrium point or some nominal motion are not necessary to study this qualitative comportment. In fact, this theory nicely complements Lyapounov theory by providing sufficient conditions for incremental exponential stability of smooth dynamic systems. However, an interesting relation between Lyapounov approach and contraction theory is given in [5]: in this paper, authors suggest to consider Lyapounov functions as metrics between two points of the state space and shown that the integral approach of the flows trajectories is needed to define the distance between two distinct points. Furthermore, energy between two infinitesimal close trajectories can be defined as a differentiable function depending on time and space together[2] and contraction behavior results from this energy decrease.

Historically, ideas closely related to contraction theory can be traced back to the middle of the last century: in [27], Hartman obtained contraction results by considering a Riemannian metric (in function of the state space only). In [51], Opial studied asymptotic stability without taking into account existence of an equilibrium point and dealt only with close trajectories in the attractive domain, the obtained results are very close to those obtained for general continuous systems. In [43], Lewis considered a Finsler metric (that is in function of the time, the state space and the velocity), it is more general than Riemannian metric (that is only in function of time and state space), he gave a particular distance estimation between two close trajectories. Furthermore, in a previous paper [42], Lewis used a particular Finsler metric depending on state space and velocity to propose some exponential convergence results be-

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[1] This means that any two points of $K$ can be relied by a continous path [3].

[2] The energy is not uniform and this is discussed[29].