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ANALYSIS OF REAL VERHULST NETWORKS

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Abstract. We investigate the dynamics of a network of scalar Verhulst systems connected in arbitrary topology, also referred to as Real Verhulst Networks. The novel contributions of this paper are the mathematical proofs for the existence of convergent stable equilibrium states and the experimental exploration of bifurcation patterns in Real Verhulst Networks. We also explore the real world implications from the behaviors exhibited by the system. **Keywords.** Verhulst, Network, Non-linear dynamics, population

1 Introduction

The Verhulst equation, also known as logistic equation, was coined by Verhulst [7], [8] to model population growth in a region . The discrete-time real Verhulst equation is:

$$x[n+1] = ax[n]\left(1 - \frac{x[n]}{K}\right) \tag{1}$$

where a is the rate of growth of population and K is the carrying capacity of the region.
A recent development is the Grey Verhulst Model, which enables one to use training data from the real world to form a Verhulst model in order to to predict the future course of various saturated S-form sequences such as Network Security Situations [3] and the number of students taking entrance examinations [4]. This is even extended to neural networks in which Grey Verhulst Models are used to complement back propagation algorithms in time-series forecasting [9].

This paper does not involve the use of Grey Verhulst models for predicting various phenomenon using real world data. Instead, we return to the original concept of a scalar Verhulst system as described by the discrete logistic equation [2] [7] [8] and we extend it over multiple regions, each with its own rate and carrying capacity. A Real Verhulst Network (RVN) is a network of such regions connected by weights. Each node in the network represents a region. Each weight represents the percentage of total population migrating from one region to another. The state of a node at the instance n represents the population of that region in a discrete time duration (1 year, 10 years, etc.). Any RVN with N nodes can be characterised by the following equation:

$$x_i[n+1] = a_i \sum_{j=1}^{N} w_{ij} x_j[n] \left(1 - \frac{\sum_{k=1}^{N} w_{ik} x_k[n]}{K_i}\right) \tag{2}$$

where $i \in \{1, 2, .., N\}$.
Here, a_i is the rate of population growth for region i, K_i is the carrying capacity for region i, $x_i[n]$ is the state of node i at time n, w_{ij} is the fraction of population in region j that is immigrating to region i and w_{ii} is the fraction of population in region i that does not immigrate.
The Verhulst Network is *real* because the rates and populations of every node is real as opposed to a complex Verhulst network, which has complex rates and node values. *This paper mathematically and experimentally analyzes the dynamics of RVNs.*
In this article, the main assumptions about any RVNs is:

- It is in discrete time.
- The fraction of population moving from one region to another is always non-negative.
- The updation of population (or state updation) at all nodes is done in parallel.
- If the population post-immigration exceeds carrying capacity, then population at the node becomes zero. In equation terms, if $\sum_{k=1}^{N} w_{ik} x_k[n] > K_i$, then $x_i[n+1] = 0$.

2 Essential operations for analyzing RVNs

2.1 Solving for equilibrium points using Eigenvectors

Let the weight matrix be $\bar{W}$.
If $\bar{X} = [X_1, X_2, ..., X_N]^T$ is an eigenvector of $\bar{W}$, then $\bar{W}\bar{X} = \lambda\bar{X}$ where λ is the eigenvalue. If $\bar{X}$ is an eigenvector, then $\alpha\bar{X}$ is also an eigenvector. Therefore,

$$\bar{W}\left(\alpha\bar{X}\right) = \lambda\left(\alpha\bar{X}\right) \tag{3}$$

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