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UNBOUNDED ELLIPTIC EQUATION WITH SINGULAR CRITICAL GROWTH TO THE GRADIENT

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Abstract. We are interested in the existence of positive solutions for unbounded elliptic problem with singular critical growth to the gradient of the form

$$\begin{cases} -div[(1+u^q)\nabla u] + \frac{|\nabla u|^2}{u^\theta} = f & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 3$, $q > 0$. For a given θ in $[1, 2)$ and with less regularity on the source term, we establish an existence and regularity results of solutions.

Keywords. Quasilinear elliptic equations, Unbounded elliptic equations, critical growth, singular equations, weak solutions.

1 Introduction

We deal in this paper with the existence of solutions for elliptic problem with singular dependence with respect to the solution. The problem is the following one:

$$\begin{cases} -div[(1+u^q)\nabla u] + b(x)\frac{|\nabla u|^2}{u^\theta} = f & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open, bounded subset of $\mathbb{R}^N$, $(N \geq 3)$ and $q > 0$. We assume that

$$1 \leq \theta < 2, \tag{2}$$

and that $b(x)$ are measurable function such that

$$0 < \alpha \leq b(x) \leq \beta. \tag{3}$$

Moreover f is a positive function belonging to some Lebesgue space $L^m(\Omega)$, with $m \geq 1$.

A possible motivation for the study of these problems arises from the Calculus of Variations. If $0 \leq f \in L^m(\Omega)$, $m > \frac{N}{2}$ and $r \in (0, 1)$, a purely formal computation shows that the EulerLagrange equation associated to the functional

$$J(v) = \frac{1}{2}\int_\Omega (1+|u|^{1-r})|\nabla u|^2 - \int_\Omega fv,$$

is

$$-div[(1+|u|^{1-r})\nabla u] + \frac{1-r}{2}\frac{u}{|u|^{1+r}}|\nabla u|^2 = f.$$

As is well known, existence and regularity of the solution for the following problem

$$\begin{cases} -\mathrm{div}(M(x,u)\nabla u) + g(x,u)|\nabla u|^2 = f(x) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{4}$$

was obtained by Boccardo, Murat and Puel [6], Bensoussan, Boccardo and Murat [2] and Boccardo, Gallout [4], where g is nonsingular, that is g is a Caratheodory function on $\Omega \times [0, \infty)$ with data f in suitable Lebesgue spaces and $M(x, u)$ is a Caratheodory bounded function subject to certain structural inequalities. The study of these problem arises from the calculus of variations if $0 \leq f \in L^m(\Omega)$ with $m \geq N/2$ and

$$g(x,u) = \frac{1}{u^\theta}, \tag{5}$$

where $\theta \in (0, 1)$. Which is considered by J-P. Puel in [10].

Let us note that in [3] the authors considered problem (4) with

$$g(x,u) = \frac{Q(x,u)}{u^\theta}, \tag{6}$$

where $\theta \in (0, 1)$ and $M(x, u)$ is a bounded function subject to certain structural inequalities.

Moreover, we refer to [5] where shown the existence and regularity of positive solutions to the differential problem

$$\begin{cases} -div[(a(x)+u^q)\nabla u] + b(x)\frac{1}{u^\theta}|\nabla u|^2 = f & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{7}$$

depending on the values of $q > 0$, $0 < \theta < 1$, and on the summability of the datum $f \geq 0$ in Lebesgue spaces.

The purpose of this paper is to focus our attention on problem (1) in the case where $1 \leq \theta < 2$ and $M(x, u)$

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